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Advanced Linear Algebra (MA 409)
Problem Sheet - 27

Orthogonal Projections and the Spectral Theorem

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1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

   (a) All projections are self-adjoint.

   (b) An orthogonal projection is uniquely determined by its range.

   (c) Every self-adjoint operator is a linear combination of orthogonal projections.

   (d) If T is a projection on W, then $T(x)$ is the vector in W that is closest to x.

   (e) Every orthogonal projection is a unitary operator.

2. Let $V = \mathbb{R}^2$, $W = span(\{(l, 2)\})$, and β be the standard ordered basis for V. Compute $[T]_\beta$, where T is the orthogonal projection of V on W. Do the same for $V = \mathbb{R}^3$ and $W = span(\{(1, 0, 1)\})$.

3. For each of the following matrices A :

   a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$

   d) $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ e) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

   (a) Verify that L_A possesses a spectral decomposition.

   (b) For each eigenvalue of L_A, explicitly define the orthogonal projection on the corresponding eigenspace.

   (c) Verify your results using the spectral theorem.

4. Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection of V on W, then $I - T$ is the orthogonal projection of V on $W^\perp$.

5. Let T be a linear operator on a finite-dimensional inner product space V.

   (a) If T is an orthogonal projection, prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$. Give an example of a projection for which this inequality does not hold. What can be concluded about a projection for which the inequality is actually an equality for all $x \in V$?

   (b) Suppose that T is a projection such that $\|T(x)\| \leq \|x\|$ for $x \in V$. Prove that T is an orthogonal projection.

6. Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then T is also an orthogonal projection.

7. Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ of T to prove the following results.

   (a) If g is a polynomial, then

   $$g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i.$$

   (b) If $T^n = T_0$ for some n, then $T = T_0$.

   (c) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i.

   (d) There exists a normal operator U on V such that $U^2 = T$.

   (e) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.

   (f) T is a projection if and only if every eigenvalue of T is 1 or 0.

   (g) $T = -T^*$ if and only if every λ_i is an imaginary number.

8. We recall a Corollary of the spectral theorem that if $F = \mathbb{C}$, then T is normal iff $T^* = g(T)$ for some polynomial g. Use the Corollary to show that if T is a normal operator on a complex finite-dimensional inner product space and U is a linear operator that commutes with T, then U commutes with T^*.

9. Prove the following facts about a partial isometry U.

   (a) U^*U is an orthogonal projection on W.

   (b) $UU^*U = U$.

10. *Simultaneous diagonalization*. Let U and T be normal operators on a finite-dimensional complex inner product space V such that $TU = UT$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both T and U.

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